

Answer any **one** question from section 1, and any **four** questions from section 2.

No marks will be awarded in absence of complete justification. Notations are as used in class.

## 1 Group Theory

- State and prove *orbit-stabilizer theorem* for group actions.
  - Let  $p$  be a prime and let  $H$  be a  $p$ -subgroup of the finite group  $G$ . Using (a) or otherwise, show that
$$[N_G(H) : H] \equiv [G : H] \pmod{p}.$$
  - Let  $P$  be a  $p$ -subgroup of  $G$ . Show that  $P$  is a Sylow  $p$ -subgroup of  $G$  if and only if  $P$  is a Sylow  $p$ -subgroup of  $N_G(P)$ . (3+4+3)
- Prove that the alternating group  $A_5$  is simple.
  - Classify groups of order 2026. Are all these groups solvable? (5+5)

## 2 Field and Galois theory

- Show that if  $\Phi_n(x)$  denotes the  $n$ -th cyclotomic polynomial, then
$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$
  - Show that for every prime  $p$ , and every  $n \in \mathbb{N}$ , there exists a unique field, say  $\mathbb{F}_{p^n}$ , of order  $p^n$ , upto isomorphism. Show that  $\mathbb{F}_{p^n}/\mathbb{F}_p$  is a simple Galois extension.
  - If  $n > 2$  and  $\zeta$  is a primitive  $n$ -th root of unity over  $\mathbb{Q}$ , then show that  $[\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}] = \phi(n)/2$ , where  $\phi$  denotes the Euler- $\phi$  function. (3+4+3)
- Show that if  $K/F$  is a finite extension, then  $K$  has only finitely many intermediate fields over  $F$  if and only if  $K$  is a simple extension over  $F$ .
  - State and prove *Primitive Element Theorem* for finite separable extensions.
  - Give an example to show that finite inseparable extensions need not be simple. (4+2+4)
- Let  $L/F$  be a finite Galois extension and  $\alpha$  be algebraic over  $L$  with minimal polynomial  $f(x)$  over  $L$ . The Galois group of  $L/F$  acts on  $f(x)$  by acting on its coefficients. Prove that the minimal polynomial of  $\alpha$  over  $F$  is the product of all distinct elements  $(\sigma \cdot f)(x)$  as  $\sigma$  runs over  $\text{Gal}(L/F)$ , where  $\cdot$  denotes the action.

- (b) Let  $L/F$  be a finite Galois extension. Suppose  $f(x)$  is monic in  $F[x]$  and its coefficients generate  $L$  over  $F$ . Show that if the product of the distinct polynomials  $(\sigma \cdot f)(x)$ , as  $\sigma$  runs over  $\text{Gal}(L/K)$ , is irreducible in  $F[x]$ , then  $f(x)$  is irreducible in  $L[x]$ . (5+5)
4. (a) Show that given a finite abelian group  $G$  there exists a Galois extension  $K/\mathbb{Q}$ ,  $K$  contained in a cyclotomic extension, such that  $\text{Gal}(K/\mathbb{Q}) = G$ .
- (b) Let  $F$  be a field of characteristic  $p$ . Prove that either  $f(x) = x^p - x - c \in F[x]$  splits completely in  $F$  or is irreducible over  $F$ . Find the Galois group of  $f(x)$  over  $F$ . (7+3)
5. Fix  $n \in \mathbb{N}$ . Let  $F$  be a field such that  $\text{char}(F)$  does not divide  $n$ , and  $F$  contains all  $n$ -th roots of unity.
- (a) Show that the simple extension  $F(\sqrt[n]{a})$  for some  $a \in F$  is a cyclic extension over  $F$ , of degree dividing  $n$ .
- (b) Let  $K$  be any cyclic extension of degree  $n$  over  $F$ . Then show that  $K = F(\sqrt[n]{a})$  for some  $a \in F$ .
- (c) Find the Galois group of the polynomial  $f(x) = 2x^5 - 10x + 5$  over  $\mathbb{Q}$ . (3+5+2)

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